UMPSC & Fall 2023

problem set 1

Notes: You may use any resources to solve these problems. Make sure you understand your solution thoroughly if you submit for credit. Good luck!

Problem 1

Part a. Prove the Central Limit Theorem (for finite variance), which states that *any* random variable X with a mean μ and finite variance σ^2 will have an IID sample mean distribution that converges to a normal distribution for large sample size n:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Part b. What special properties of the normal distribution cause it to be the limiting distribution?

Problem 2

A particle is released at $(0,e) \in \mathbb{R}^2$ within the xy plane. The particle undergoes decoupled temporal motion (with respect to time t) given by the equations

$$dx = \mathcal{N}(0,1)\sqrt{dt} \qquad \text{(brownian motion)}$$

$$dy = -y\,dt$$

until it reaches the line $y = 0.1$. At this moment, the particle begins to obey the equations of motion

$$dx = \sin(\alpha x)\,dt$$

$$dy = -y\,dt$$

for some $\alpha \in \mathbb{R}$.

Let $1 >> \epsilon > 0$, and determine the probability, with respect to α, of the particle reaching a stable state $S \in \mathbb{R}^2$ where $||S-(\pi,0)|| < \epsilon$ for sufficiently large t. You may express your answer in terms of the standard normal cumulative distribution function Φ.

Problem 3

Suppose $\mu : \mathscr{B}(\mathbb{R}) \to \mathbb{Q}\cap[0,1]$ is a rational-valued probability measure on the Borel σ-algebra $\mathscr{B}$ over $\mathbb{R}$. Prove that μ takes finitely many values.